57–8

Proceedings of the American Academy of Arts and Sciences.

Vol. 57. No. 8.—May, 1922.

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By George W. Pierce.

(Continued from page 3 of cover.)

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Received Jan. 13, 1922. Presented Jan. 11, 1922.

1. General Principles.

1. Introduction.— So far as I have been able to ascertain, the artificial lines with lumped sections heretofore employed have been generally devoid of appreciable mutual inductance between adjacent elements.

In artificial line construction the inductive elements are placed mutually at right angles to each other or are wound in toroidal form so as to reduce to a minimum the mutual inductance caused by magnetic leakage from one inductive element to the next.

In my Text *Electric Oscillations and Electric Waves*,[1] Chapter XVI, I have treated theoretically artificial lines in which mutual inductance exists between adjacent series elements.

It is proposed here to show, with the aid of the general treatment in my text, that such mutual inductance, if properly chosen, has a decidedly beneficial effect in the following two types of apparatus:

A. An Electric Compensator designed to introduce into circuits a time lag substantially independent of the frequency of impressed e.m.f.;

B. An Artificial Line designed to simulate an actual smooth line.[2]

2. General Type of Line.— Let us direct our attention to the general type of artificial line shown in Figure 1.

The shunt elements of complex impedance z_2 may be of any character. The series elements of complex impedance z_1 may likewise be of any character, and have between such of these elements as are adja-

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[1] Pierce: *Electric Oscillations and Electric Waves*, McGraw-Hill Book Co., New York, 1920. (See also corrected reprint in press 1921.)

[2] Since sending this paper to press I have been informed by Mr. K. S. Johnson that application of the present device in Case B has been in use for several years at the Western Electric Company, but has not been described in publications.

cent a mutual inductance M, which may be set equal to zero in case results without M are required. The mutual inductance between non-adjacent elements is supposed to be negligible in all cases.

The line is supposed to terminate at both ends in a series half-element of complex impedance $z_1/2$ so constructed as to have also mutual inductance M with its neighboring series section.

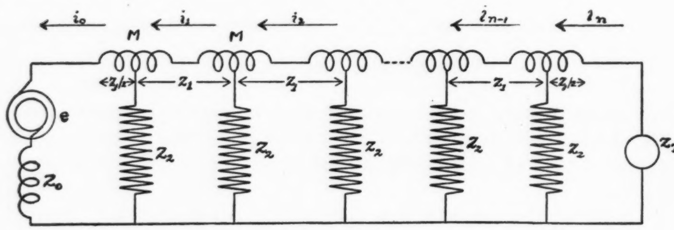

FIGURE 1. General type of artificial line containing mutual inductance between adjacent series elements.

The line is shown connected to an input terminal apparatus of complex e.m.f. e and complex impedance z_0 and connected to an output apparatus of complex impedance z_T.

The artificial line here shown may be regarded as consisting of n–1 loops and two terminal series half elements, or it may be considered to consist of n sections, known as T-sections, each of the character shown in Figure 2.

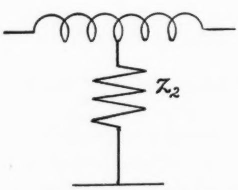

FIGURE 2. A T-section.

It may be noted that the treatment of other general types of lines (for example, lines of π-sections) are deducible from the treatment of the line of Figure 1 by postulating certain changes in the terminal conditions.

3. General Definitions.—

z_1 = *complex impedance* of the series elements.

z_2 = *complex impedance* of the shunt elements.

M = *mutual inductance* between adjacent series elements, estimated positive when the coils are wound in the same sense.

φ = *retardation angle of current* per section of the line = the angle of lag of steady-state current in any section behind the current in the preceding section.

T = *time lag of current* in seconds per section.

a = *real attenuation constant of current* per section of the line = the exponent in the factor ϵ^{-a}, by which factor the steady-state current-amplitude in any series element of an infinite or non-reflective line must be multiplied to give the current-amplitude in the next series element.

z_i = *complex surge impedance* of the line = complex impedance of a line of an infinite number of sections or a line with non-reflective output impedance.

ω = *angular velocity* of impressed sinusoidal e.m.f. in radians per second.

4. Abbreviations and Subsidiary Notation.— The following notation taken from *Electric Oscillations and Electric Waves* with slight augmentation will be employed.

$$P + jU, = \frac{z_1 + 2z_2}{z_2 - Mj\omega} \tag{1}$$

where P and U are real quantities;

$$V = \frac{1 - U^2 - P^2}{2}; \tag{2}$$

$$h = U/V; \tag{3}$$

$$f(h) = +\sqrt{\frac{+\sqrt{1 + h^2} + 1}{2}}, \quad g(h) = +\sqrt{\frac{+\sqrt{1 + h^2} - 1}{2}}. \tag{4}$$

Other notation will be introduced below in Section **10**.

5. Equations for a and φ for Steady-State Current.— It is proved in *Electric Oscillations and Electric Waves*, equations (49) and (50), p. 296, that, for the steady-state current under the action of an

impressed sinusoidal e.m.f. of constant amplitude and frequency, a and φ are given by expressions which factored may be written

$$\left.\begin{aligned} a &= \sinh^{-1}\left\{+\sqrt{\pm V}\sqrt{\sqrt{1+h^2}\mp 1}\right\}, \\ \varphi &= \sin^{-1}\left\{+\sqrt{\pm V}\sqrt{\sqrt{1+h^2}\mp 1}\right\}, \end{aligned}\right\} \tag{5}$$

in which V and h have the value given in (2) and (3), and *in which the upper signs or the lower signs are to be used together in each case and are to be chosen so as to make a and φ both real quantities.* For definitions of a and φ see §**3**.

6. Modified Form of Equations for a and φ.— Using the f- and g-functions given in (4), we may write (5) as follows:

$$\left.\begin{aligned} &\text{I. If } V > 0, \\ &\qquad a = \sinh^{-1}\sqrt{2V}\, g(h), \quad \varphi = \sin^{-1}\{\pm\sqrt{2V}\, f(h)\} \\ &\text{with} \qquad h = U/V. \\ &\text{II. If } V < 0, \\ &\qquad a = \sinh^{-1}\sqrt{-2V}\, f(h), \quad \varphi = \sin^{-1}\{\pm\sqrt{-2V}\, g(h)\} \\ &\text{with} \qquad h = U/V. \end{aligned}\right\} \tag{6}$$

These equations are to be employed subject to the following rule regarding the quadrant of φ:

Rule Regarding φ.

sign of		quadrant of φ
P	U	
+	+	first
−	+	second
−	−	third
+	−	fourth

Equations (6) are the general expressions for the real attenuation constant a and the real retardation angle φ per section in terms of U, V, $f(h)$,

and $g(h)$, which are defined in equations (1), (2), (3), and (4). These equations apply to the general type of line given in Figure 1.

In a paper now in press[1] entitled *A Table and Method of Computation of Electric Wave Propagation, Transmission Line Phenomena, Optical Refraction, and Inverse Hyperbolic Functions of a Complex Variable* I have given a table of the functions $f(h)$ and $g(h)$ for various values of h, so as to render very simple the computations of a and φ of equations (6).

7. General Equation for Surge Impedance z_i.— Before passing to a further discussion of a and φ, we shall introduce the general expression for surge impedance z_i, taken from *Electric Oscillations and Electric Waves,* Equation (34), p. 292, as follows:

$$z_i = \pm \sqrt{\frac{(z_1 + 2z_2)^2}{4} - (Mj\omega - z_2)^2}. \tag{7}$$

In Equation (7) the sign before the radical must be chosen to make the real part positive.

It may be noted that this equation also permits of easy computation by the method of the paper referred to in Section **6**.

8. Time Lag per Section.—

Let

T = time lag in seconds per section of the line introduced into the current by the line,

ω = angular velocity in radians per second of the impressed e.m.f.

In the steady state, the current will also have the angular velocity ω and the time lag per section will be given by

$$T = \varphi/\omega. \tag{8}$$

The steady-state time lag in seconds per section is the retardation angle per section in radians divided by the angular velocity in radians per second.

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[1] These Proceedings: Vol. 57, No. 7.

II. The Electric Compensator.

Improvement Introduced by Proper Mutual Induction between Series Sections.

9. Brief Description.— In the determination of the direction of submarine sound signals and in giving to submarine sound apparatus directive qualities so as to permit discrimination of certain sounds from other sounds coming from a different direction, Professor Max Mason [1] of Wisconsin University has made use of two or more sound detectors (rubber nipples in the water) communicating with the ear of the observer through paths (air pipes) capable of adjustment as to time of travel (by adjusting pipe lengths), so that, when the detectors are struck, one after the other, by a sound-wave front, the impulses set up in the transmission paths connected to the several receivers may all be brought to the ear together by a suitable adjustment of retardation by the paths (compensation). In this way the setting of the apparatus (a compensator) to give a maximum of sound will, when the apparatus is properly calibrated, give a direct reading of the direction of the incident sound. Sounds coming from other directions (as, for example, noises from the listener's boat) will in general not be compensated to give a maximum of intensity and will be discriminated against by the apparatus.

Professor Mason developed this apparatus in a form known as the *Accoustical System* employing rubber nipples on the ends of tubes as sound detectors, and introducing compensation by varying the lengths of air colums through which the resulting sound waves in the air columns were transmitted to the ear of the observer. The Mason system gives excellent results in practice.

It readily occurred to those who knew of Mason's Device that some advantage might result by using microphones, or other electrical detectors, in the place of the rubber nipples, provided electrical methods could be employed to produce the required retardations of currents set up at the microphones.

An apparatus for this purpose was devised by me at the Naval Experimental Station at New London, and is called an *Electric Compensator.*

An Electric compensator is a device for giving electrically to an electric

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[1] See H. C. Hayes, *Detection of Submarines*, Proc. Am. Phil. Soc., Vol. 59, pp. 1–47, 1920: and *U. S. Navy MV-Type of Hydrophone as an Aid and Safeguard to Navigation*, Ibid., Vol. 59, pp. 371–404; Max Mason: *Submarine Detection by Multiple Unit Hydrophones*, Wisconsin Engineer (1921).

current any desired time-retardation substantially independent of the frequency over a significant range of frequencies.

An artificial line consisting of inductances in series and capacities in shunt and provided with suitable switching devices is an electric compensator. To have practical application the attenuation of the compensator must be low and the steps must be made small enough in inductance and capacity values to obviate filtering effects, and the dimensions of the coils must be chosen in a manner to diminish dispersion and other distortions. The principles governing the design of such a compensator are given in my book *Electric Oscillations and Electric Waves*, pp. 285–323.

It is proposed here to show how improvements result from the use of suitable mutual inductance between the neighboring series inductances of the electric compensator.

10. Proof that Introduction between neighboring sections of Mutual Inductance Approximately Equal to One-Tenth of the Self Inductance of the Series Elements Greatly Reduces Dispersion in an Electric Compensator.— It is proposed to prove

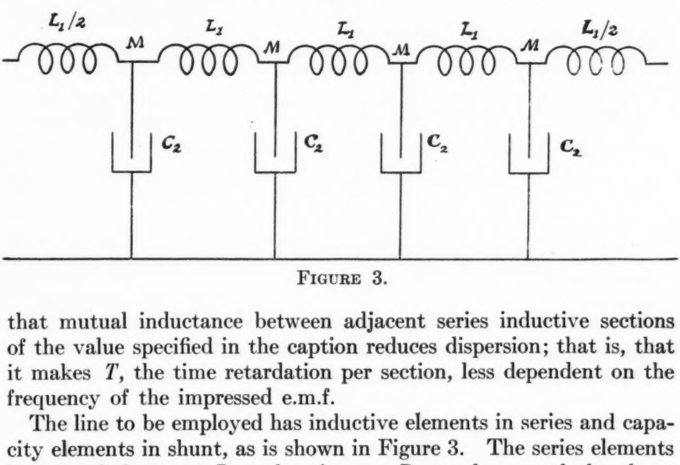

FIGURE 3.

that mutual inductance between adjacent series inductive sections of the value specified in the caption reduces dispersion; that is, that it makes T, the time retardation per section, less dependent on the frequency of the impressed e.m.f.

The line to be employed has inductive elements in series and capacity elements in shunt, as is shown in Figure 3. The series elements have an inductance L_1 and resistance R_1 per loop, and the shunt elements have each a capacity C_1. Let there be mutual inductance M between the inductances of each two adjacent loops, and let there be zero mutual inductance between inductances not adjacent.

We shall then have (compare *El. Osc. and El. Waves*, p. 316)

$$z_1 = R_1 + jL_1\omega, \tag{9}$$

$$z_2 = - j/C_2\omega, \tag{10}$$

These values substituted into (1) give

$$U = \eta Q/2, \tag{11}$$

$$V = \frac{Q}{2}\left\{1 - \frac{Q}{4}(1 + \eta^2)\right\}, \tag{12}$$

in which the following abbreviations are employed

$$L = L_1 + 2M, \qquad R = R_1, \tag{13}$$

$$\eta = R/L\omega, \quad \text{and} \quad Q = \frac{LC_2\omega^2}{1 + MC_2\omega^2}. \tag{14}$$

As a further abbreviation, let us write

$$A = 1 - \frac{Q}{4}(1 + \eta^2), \tag{15}$$

so that equation (12) becomes

$$V = QA/2. \tag{16}$$

Then by the general equations (6), we have for φ the value

$$\text{if} \qquad V > 0, \quad \varphi = \sin^{-1}\sqrt{\frac{QA}{2}}\sqrt{\sqrt{1 + \frac{\eta^2}{A^2}} + 1}\,. \tag{17}$$

This equation is the equivalent of (142), p. 317, of *Electrical Oscillations and Electric Waves.* We now proceed to develop the subject further.

Confining our attention to this case in which V is greater than zero (that is, A positive), and expanding the inner radical in (17), we obtain, after transposition and squaring,

$$\sin^2\varphi = QA\left\{1 + \frac{\eta^2}{4A^2} - \frac{\eta^4}{16A^4} + \cdots\right\} \tag{18}$$

If now we consider the resistance per section to be so small in comparison with the inductive reactance per section that

$$\eta^2 << 1 \text{ and } \eta^2 << 4A^2 \tag{19}$$

we have

$$\sin^2\varphi = QA = Q - Q^2/4. \tag{20}$$

Let us now expand the two sides of (20) in series, obtaining

$$\varphi^2 - \frac{\varphi^4}{3} + \frac{2\varphi^6}{45} - \cdots = LC_2\omega^2 (1 - MC_2\omega^2 + M^2C_2{}^2\omega^4 - \cdots)$$
$$- \frac{L^2C_2{}^2\omega^4}{4}(1 - 2MC_2\omega^2 + 3M^2C_2{}^2\omega^4 - \cdots). \tag{21}$$

By reference to (8) it will be seen that to make T, the time retardation per section, independent of the frequency, we shall require φ^2 to be proportional to ω^2. This requirement will be consistent with (21) provided we can so choose M that

$$\varphi^2 = LC_2\omega^2, \tag{22}$$

and

$$\frac{-\varphi^4}{3} + \frac{2\varphi^6}{45} = -\left(\frac{L^2C_2{}^2\omega^4}{4} + \frac{LMC_2{}^2\omega^4}{1}\right) + \left(\frac{L^2MC_2{}^3\omega^6}{2} + \frac{LM^2C_2{}^3\omega^6}{1}\right) \tag{23}$$

are both satisfied.

Equations (22) and (23) are equations from which to determine M. Instead of an exact determination of M we shall content ourselves by an approximate determination of M by substituting (22) into (23) and equating separately terms of the same power of ω, obtaining

$$\frac{L^2C_2{}^2\omega^4}{12} = LMC_2{}^2\omega^4, \text{ and} \tag{24}$$

$$\frac{2L^3C_2{}^3\omega^6}{45} = \frac{L^2MC_2{}^3\omega^6}{2} + LM^2C_2{}^3\omega^6. \tag{25}$$

Equation (24) gives

$$M = L/12, \tag{26}$$

which substituted into (25) gives

$$L^3/45 = L^3/41. \tag{27}$$

Equation (27) is not true, but since terms in which L^3 enters are of a higher order than terms in L and L^2, we may consider the difference between the two sides of equation (27) as negligible, and employ (26) as the approximate condition required.

By the use of (13), this condition (26) becomes

$$M = L_1/10, \tag{28}$$

and by (8) and (28) equation (22) gives

$$T = \sqrt{LC_2} = \sqrt{1.2L_1C_2}. \tag{29}$$

Equation (28) gives the value of the Mutual Inductance M to be introduced between the adjacent series coils of Self Inductance L_1 (and of negligible resistance) in order to make the time-retardation T per section be as nearly as possible independent of the frequency and to have over a wide range of frequencies approximately the value given in (29).

If the resistance of the coils is not negligible it is possible to modify the relation (28) slightly in a manner that will enhance slightly the constancy of T over a given range of frequencies, but in practice, where a reasonable effort is made to keep resistances small, the relation (28) is sufficiently accurate.

In any case the exact value of T can be calculated by the use of (17) and (8).

11. Computed Table and Curves Showing the Performance of a Line with Mutual Inductance Equal to One Tenth of Self Inductance in Comparison with a Line of Zero Mutual Inductance.— Table I of which four columns were computed from the exact Equation (17) with the use of (8), gives the values of $T/\sqrt{LC_2}$ for different values of $\omega\sqrt{LC_2}$.

TABLE I.

Values of $T/\sqrt{LC_2}$ for Various Values of $\omega\sqrt{LC_2}$ for the Case of $R/R_0 = 0$ and the Case of $R/R_0 = .1$, with $M = 0$ and $M = .1L_1$ in each case.

	$R/R_0 = 0$		$R/R_0 = .1$			
	$M = 0$, $L = L_1$	$M = .1L_1$, $L = 1.2L_1$	$M = 0$, $L = L_1$	$M = .1L_1$, $L = 1.2L_1$	$M = 0$, $L = L_1$	$M = .1L_1$, $L = 1.2L_1$
$\omega\sqrt{LC_2}$	$T/\sqrt{LC_2}$	$T/\sqrt{LC_2}$	$T/\sqrt{LC_2}$	$T/\sqrt{LC_2}$	a	a
.00	1.0000	1.0000	∞	∞	.00000	.00000
.01	1.0000	1.0000	2.347	2.347	.02131	.02131
.02	1.0000	1.0000	1.745	1.745	.0287	.0286
.03	1.0000	1.0000	1.497	1.497	.0334	.0334
.05	1.0000	1.0000	1.272	1.270	.0394	.0394
.10	1.0007	1.0000	1.099	1.098	.0456	.0456
.15	1.0017	1.0000	1.050	1.049	.0479	.0478
.20	1.002	1.000	1.030	1.029	.0489	.0487
.25	1.003	1.000	1.022	1.019	.0495	.0494
.30	1.004	1.000	1.017	1.013	.0500	.0498
.35	1.005	1.000	1.016	1.009	.0503	.0499
.40	1.007	1.000	1.015	1.007	.0507	.0503
.50	1.011	1.000	1.016	1.005	.0514	.0508
.60	1.016	1.000	1.018	1.004	.0523	.0514
.70	1.022	1.001	1.023	1.002	.0532	.0521
.80	1.032	1.001	1.031	1.003	.0545	.0528
1.00	1.050	1.002	1.046	1.004	.0577	.0547
1.20	1.075	1.004	1.072	1.007	.0624	.0574
1.40	1.121	1.009	1.109	1.010	.0700	.0605
1.60	1.156	1.019	1.160	1.016	.0834	.0659
1.80	1.239	1.027	1.241	1.026	.1142	.0736
2.00	1.570	1.047		1.044		.0862
2.20		1.082		1.079		.1129
2.40		1.172		1.144		.2122

The last two columns contain corresponding values of the attenuation constant a computed by the use of the exact equation (6), for the case of $R/R_0 = .1$. For $R = 0$ the value of a is zero throughout the range of the table.

In the table the values of R are specified by specifying values of R/R_0, where $R_0 = \sqrt{L/C_2}$. Only two values of R/R_0 (namely, zero and .1) were employed in computing the tables. The case in which the ratio is zero is a case of zero attenuation, while the case with the ratio .1 is a case of much higher attenuation than would arise with coils built with a reasonable effort to keep the ratio of resistance to inductance small, so that the table gives extreme values.

Curves of the data of the table are presented in Figures 4, 5 and 6.

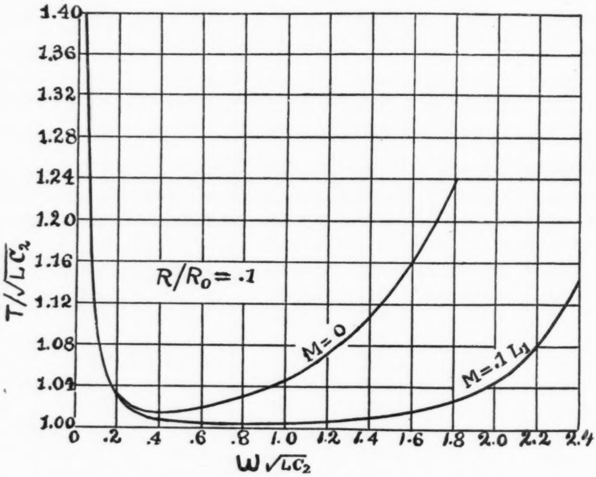

FIGURE 4.

12. Comparison of Line of Mutual Inductance equal to One Tenth of Self Inductance with Line of Zero Mutual.— By reference to the curves of Figures 4 and 5 it is seen that the range of values of ω for which the time retardation T per section is within a given percentage of constant is much larger in the case $M = .1L_1$ than in the case $M = 0$. With $M = .1L_1$ the range of frequencies over which T is within one percent of constant is three times as great as when no

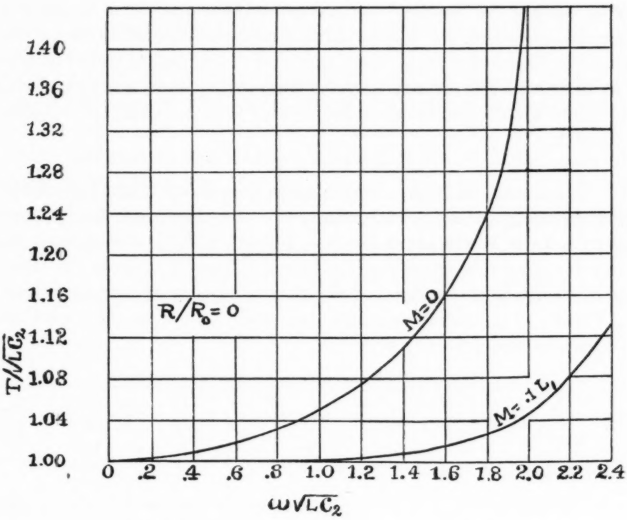

FIGURE 5.

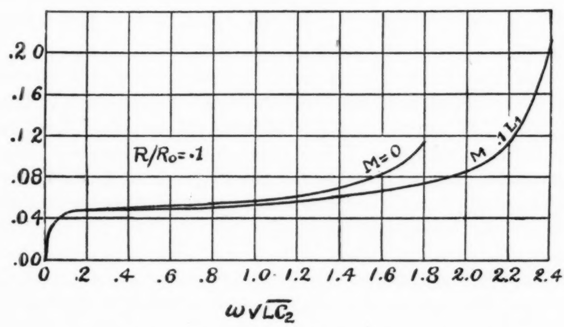

FIGURE 6.

mutual inductance is introduced between the series coils. Reference to Figure 6 shows that the attenuation constant a' is also slightly smaller with the mutual inductance than without it.

If air-core coils are to be used it requires less space and is much

easier to supply the required mutual inductance than it is to leave it out, so that by using the mutual inductance between sections we can increase the frequency range for a specified constancy of time retardation to three times the range without the mutual inductance, and this can be done with a better space factor and with a significant diminution of cost.

13. Constancy of Surge Impedance also Somewhat Improved by Introducing Mutual Inductance = $.1L_2$ between Series Sections of a Low Resistance Line.— Equation (7) is the general expression for surge impedance of the line of Figure 1. When the series impedances are inductances and the shunt impedances are capacities, as in Figure 3, z_1 and z_2 take on the values given in (9) and (10). These substituted into (7) give

$$z_i = \pm\sqrt{\frac{L}{C_2}}\sqrt{1+\left(\frac{R}{2R_0}\right)^2-\frac{L-4M}{4}C_2\omega^2+j\left\{\frac{R}{2R_0}\sqrt{LC_2}\omega\left(1-\frac{2M}{L}\right)-\frac{R}{R_0}\frac{1}{\sqrt{LC_2}\omega}\right\}}. \quad (30)$$

Equation (30)[1] *is the general expression for surge impedance of a line of the type shown in Figure 3. In this equation $L = L_1 + 2M$, and $R_0 = \sqrt{L/C_2}$.*

It is seen that, if the resistance of the line is low, the imaginary term in (30) is small, and the real term tends to approach independence of ω as $4M$ approaches L. Making $M = .1L_1 = L/12$, as is required in order to make T less dependent on frequency, has the effect of cutting the real term containing ω^2 by about $\frac{1}{3}$ and hence the introduction of $M = .1L_1$ reduces the dependence of surge impedance on frequency for low resistance lines.

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[1] The corresponding equation (142) p. 317 of *El. Osc. and Waves* has in the first printing of the book an error, that has been corrected in the Second Impression of 1921.

III. An Artificial Line to Simulate an Actual Smooth Line.

Improvement as to Attenuation and Surge Impedance brought about by Proper Mutual Induction between Series Sections.

14. Constants for Actual Smooth Line.— In the case of an actual smooth line, let

r, l, c = respectively the resistance, self-inductance, and capacity per loop unit of length;
α = real attenuation constant per unit of length;
β = angle of lag of current per unit of length;
z_i = surge impedance of the line;

then

$$\alpha = \omega \sqrt{\frac{lc}{2}} \sqrt{\sqrt{\frac{r^2}{l^2\omega^2} + 1} - 1} \qquad (31)$$

$$\beta = \omega \sqrt{\frac{lc}{2}} \sqrt{\sqrt{\frac{r^2}{l^2\omega^2} + 1} + 1} \qquad (32)$$

$$z_i = \pm \sqrt{\frac{l}{c}} \sqrt{1 - j\frac{r}{l\omega}}\,. \qquad (33)$$

These equations are the familiar expressions. See, for example *El. Osc. and El. Waves*, pp. 329 and 330.

15. To Design an Artificial Line with Lumped Sections that Will Simulate the Smooth Line as to Surge Impedance.— For this purpose we shall postulate a line of the type shown in Figure 3, and shall determine what value of M brings the surge impedance of this line, as is given in (30), most nearly into the form of (33) in so far as dependence of z_i on ω is concerned. Since the real part of (30) is generally much larger than the imaginary part, it is seen by inspection that a good approximation to this result is made by making

$$M = L/4; \qquad (34)$$

which by (13) means

$$M = L_1/2. \qquad (35)$$

By making M have the value given in (34) equation (30), in view of the fact that

$$R_0 = \sqrt{L/C_2}, \tag{36}$$

becomes

$$z_i = \pm\sqrt{\frac{L}{C_2}}\sqrt{1 + \frac{R^2C^2}{2L} - j\frac{R}{L\omega}\left\{1 - \frac{LC_2\omega^2}{4}\right\}}. \tag{37}$$

Equation (37) gives the surge impedance of a lumpy artificial line with $M = L_1/2$. Since the imaginary term in the surge impedance of a smooth-line (equation (33)) is generally small over practical ranges of frequency, it is seen that (37) is essentially of the form of (30) in so far as concerns dependence of surge impedance on frequency.[1]

16. To Determine the Mutual Inductance between Series Section in a Lumpy Artificial Line to Bring it into Close Similarity with A Smooth Line as to Attenuation Constant.—Postulating a line of the type of Figure 3, and substituting (9), (10), (13), (14) and (15) into the value of a given in (5) we obtain

If $V > 0$,

$$a = \sinh^{-1}\sqrt{\frac{QA}{2}}\sqrt{\sqrt{1 + \frac{\eta^2}{A^2}} - 1}. \tag{38}$$

Equation (38) is the general equation for attenuation constant for a line of the type of Figure 3, under the condition $V > 0$.

Equation (38) expanded with neglect of higher powers of η^2/A^2 gives

$$\sinh a = \frac{\eta}{2}\sqrt{\frac{Q}{A}}\left\{1 - \frac{\eta^2}{8A^2} + \frac{\eta}{128}\frac{\eta^4}{A^4} + \cdots\right\} \tag{39}$$

A corresponding expansion of (31) gives approximately for the smooth line

$$a = \frac{\eta_0}{2}\sqrt{lc}\,\omega\left\{1 - \frac{\eta_0^2}{8} + \cdots\right\}, \tag{40}$$

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1 This fact was called to my attention by Mr. Phillip Machanik, who based his observation in an examination of the equations in *Electric Oscillations and Electric Waves*.

where

$$\eta_0 = r/l\omega. \tag{41}$$

Since the second terms in (40) and (39) are usually small, and since a is also sufficiently small to make sinh a essentially equal to a, the two equations reduce to nearly the same form in respect to ω if

$$Q/A = LC_2\omega^2. \tag{42}$$

Replacing A in (42) by its value from (14) with neglect of η^2 in (15) we obtain

$$Q = \frac{LC_2\omega^2}{1 + LC_2\omega^2/4},$$

which compared with (14) shows that (42) is approximately satisfied when

$$M = L/4 = L_1/2. \tag{42}$$

A substitution of this value of M into the exact equation (38) gives in a careful approximation

$$\sinh a = \frac{\eta}{2}\sqrt{LC_2}\,\omega\left\{1 - \frac{\eta^2}{8} + \frac{\eta^2}{16}\left(LC_2\omega^2 + \frac{3}{8}L^2C_2^2\omega^4\right)\right\}. \tag{43}$$

If now we expand the hyperbolic sine into

$$\sinh a = a + a^3/6,$$

and replace the a in a^3 by the first term of (43), we obtain

$$a = \frac{\eta}{2}\sqrt{LC_2}\,\omega\left\{1 - \frac{\eta^2}{8} + \frac{\eta^2}{96}\left(LC_2\omega^2 + \frac{9}{4}L^2C_2^2\omega^4\right)\right\}. \tag{44}$$

The approximation of equation (44) shows that even when η is as large as .5 and with $LC_2\omega^2$ as large as 1 the introduction of a mutual inductance of the value given in (42) makes the attenuation constant of the artificial lumpy line of the same form as the attenuation constant α of the smooth line, and that the two attenuation constants can be thus made to agree over a wide range of frequencies.

IV. Conclusions.

The following results, believed to be novel, for artificial line construction have been here derived.

1. To obtain a minimum dependence of time lag per section on frequency of an electric artificial line of low resistance, the line should be constructed with mutual inductance between neighboring series inductive elements equal to one-tenth of the self inductance of each series inductive element.

2. To most closely simulate a real smooth electric line as to attenuation constant and surge impedance by the use of an artificial electric line with lumpy sections, there should be in the artificial line a mutual inductance between adjacent loops equal to one-half of the series self inductance per loop.

3. Details for calculating the performance of lines constructed according to 1. and 2. are given.

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